Weak Solutions to Semilinear Elliptic Equations with Nonlocal Boundary Conditions of Dirichlet Type

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We discuss existence results for semilinear elliptic equations subject to nonlocal Dirichlet-type boundary conditions. We first consider the auxiliary problem

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f\big(\boldsymbol{x}, u(\boldsymbol{x})\big) & \text{for } \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) = \displaystyle\int_\Omega K(\boldsymbol{x}, \boldsymbol{\xi}) h\big(\boldsymbol{\xi}, u(\boldsymbol{\xi})\big)\, d\boldsymbol{\xi} & \text{for } \boldsymbol{x} \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain with Lipschitz boundary and the nonlinearities f, h, together with the kernel K, satisfy suitable growth and regularity assumptions. The solvability of this problem is obtained in a weak formulation.

We then use approximation methods to study a pointwise nonlocal boundary condition of the form

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f\big(\boldsymbol{x}, u(\boldsymbol{x})\big) & \text{for } \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) = \alpha(\boldsymbol{x}) g\big(u(\boldsymbol{p}_1), \ldots, u(\boldsymbol{p}_m)\big) & \text{for } \boldsymbol{x} \in \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$ is a bounded domain with boundary of class C^2, the points $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m$ belong to Ω, the function α is Lipschitz continuous, and g is continuous with controlled sublinear growth. Particular attention is paid to the case in which the boundary condition depends on the value of the solution at a single interior point. The argument is then extended to the multipoint case.

Related existence results have previously been established mainly in spaces of continuous functions, with Krasnosel'skii's fixed point theorem in a cone as the principal tool. In contrast, the present approach treats solutions in the weak sense and relies on a hybrid fixed point argument involving the theory of matrices convergent to zero.

References

[1] M. Bełdziński, M. Galewski, I. Kossowski, *Existence results for semi-linear differential equations with nonlocal boundary value conditions*, Topological Methods in Nonlinear Analysis, 67(1), 143–163 (2026)

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