Computer-assisted proof for radial solutions of elliptic equations with Coulomb and Hardy potentials

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We are interested in radial symmetric solutions of the equation involving Coulomb and Hardy potentials:

$$-\Delta u(x) + \frac{c}{|x|^2}u(x) + \frac{a}{|x|}u(x) + bu(x) + d\big(u(x)\big)^3 = 0, \quad x \in \mathbb{R}^3,$$

where $a < 0$, $b > 0$, $d < 0$ and $c \in \mathbb{R}$. We transform our problem to the following ODE

$$-v''(r) - \frac{2}{r}v'(r) + \frac{c}{r^2}v(r) + \frac{a}{r}v(r) + bv(r) + d\big(v(r)\big)^3 = 0, \quad r \in [0, +\infty), \tag{1}$$

and require that solutions posses the asymptotic behaviour

$$\lim_{r\to+\infty} v(r) = \lim_{r\to+\infty} v'(r) = 0. \tag{2}$$

We follow the strategy from [1] and divide interval $[0, +\infty)$ into three subintervals. On the first interval $[0, l_1]$ we call for the solution given by the Taylor-type two-indices series obtaining recurrence relations involving coefficients of this series. On the third interval $[l_2, +\infty)$ we assume that the constructed solution belongs to the center-stable manifold generated by the first order system derived from (1). To show the existence of such manifold we apply the method of the Lyapunov-Perron operator (see [2, Sec. 4.1]). In addition, solution belonging to the center-stable manifold satisfies the asymptotic behaviour (2). In order to connect smoothly part of the solution obtained by the Taylor-type series and those lying in the center-stable manifold we call on the interval $[l_1, l_2]$ for the solution given by the Chebyshev series. This leads to the zero-finding problem of a map defined on a Banach space of coefficients. To find zeros we apply some variant of the Newton-Kantorovich theorem and rigorous numerics based on the interval arithmetic.

References

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[2] C. Chicone, *Ordinary differential equations with applications. 2nd ed.*, Texts Appl. Math. 34, xx, 636 pp., Springer, New York (2006)

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